An Expository Account of Dynamical Systems and Invariant Measure

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1. Introduction

The qualitative study of differential equation can be seen in the work of Henry Poincare, whose interest was to look at the problems in celestial mechanics and thereby understand the stability of solar system (model as an n-body problem) in a grand fashion. As a result, dynamical system took birth and in the present context, it means a one parameter group of transformation of space, sometimes called a phase space with the parameter representing time (which would be continuous or discrete). The theory focuses on the long-term behavior of the trajectories of points subject to under transformation. In the 1950's, Kolmogorov and Sinai characterized followed by Smale in 1960's on hyperbolic dynamics. In 1970's, new outlook and new challenges encountered. With the aid of computers, the research could come up with abundant examples whose dynamics were dominated with dynamical systems, providing those examples, characterizing flows, and understanding the stability of related issues associated with them. We are going to survey some of these developments under Anosov diffeomorphism which refers to the map which are hyperbolic on the entire manifold under consideration and axiom A where such flows are satisfied with maps that a hyperbolic uncertain essential parts of the manifold.

In the second section, we have given a brief of account of invariant measure (measure rigidity) associated with the flows on dynamical system by providing simple examples. The exposition assumes some familiarity with measure and probability measure on a space (some of this definition and examples are mentioned in the introduction for clarity). The third section relates to the notion of stability. Here, we consider system of ordinary differential equations for a function f defined $\mathbb{C}^n$ and taking values to $\mathbb{C}^n$ with fixed point at the origin. Around the fixed point, evolution of the system is studied. In the fourth section of this exposition, we have considered an interesting connection between number theory and symbolic dynamics that were already seen from the light of Diophantine approximations for the sets which are characterize as badly approximable sets for geodesic flows and their associated orbits. To conclude, some open problems and future directions are cited. The exposition is some sort of

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survey that reflects the developments the modern theory of dynamical systems and their applications.

Example 1.1 (Rotations of a Circle)**.** *We will first look at a circle, often considered as a one dimensional torus $\mathbb{T}^1$. The circle $\mathbb{S}^1$ can be presented as*

$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\} = \{z = e^{2\pi/x} : x \in \mathbb{R}\}.$$

In additive notation,

$$\mathbb{S}^1 = \mathbb{Z}\backslash\mathbb{R}.$$

These two modules are isomorphic to each other by the logarithm map $z = e^{2\pi/x} \mapsto x \pmod 1$.

Let R_θ denote the rotation of $\mathbb{S}^1$ by angle $2\pi\theta$. The map R_θ is given by

$$R_\theta(z) = ze^{2\pi i\theta} \text{ and } R_\theta(x) = x + \theta \pmod{\mathbb{Z}}$$

in multiplicative and additive models of $\mathbb{S}^1$. The orbit of $z = e^{2\pi ix} \in \mathbb{S}^1$ under iterations of R_θ is respectively equal to

$$\{ze^{2\pi in\theta} : n \in \mathbb{Z} \text{ and } \{x + n\theta \pmod 1 : n \in \mathbb{Z}\}.$$

The above example can be realized for higher dimensional torus.

1.1 Generalities:

We gather some definitions, familiar results from analysis, algebra and geometry with examples that find their place in our study that fall under the scope of dynamical systems. The underlying set is either a group, topological space, and more so vector spaces with functions defined on them and taking values again in them. Number sets like $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ occur in our discourse which enjoy good mathematical properties to strengthen our understanding of abstract association with the topological, geometric or algebraic entities.

By a ***group***, we mean a non-empty set G with a product map $P : GG \to G$ satisfying the following properties:

- P is closed; i.e., $g, g' \in G \implies P(g, g') = gg' \in G$.
- P is associative, i.e., $g(g'g'') = (gg')g'' \forall g, g', g'' \in G$.
- If g is in G then there is an element hg in G such that $g * hg = hg * g = g$; hg constitutes an identity element in G, we denote it by e. This is true for all elements g in G.
- For each g in G , there exists g' in G such that $gg' = g'g = e$. The element g' (also written as g^{-1}) is called the inverse of g in G. Note that e, the inverse elements g^{-1} for a given g, are unique in G.

The product gg' need not be equal to $g'g$. If, $gg' = g'g$ for all $g, g' \in G$ then we say that the group G is an abelian (or commutative) group.

The product operation P in G is an abstract operation. If $P(gg') = g + g'$ for $g, g' \in G$ then we say that the product operation is an addition. Familiar examples of groups include, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ under additive operation P. The set $\{1, -1\}$ under product multiplication forms a group, and $\{1\}$ under multiplication forms a group (called a trivial group).

From the geometric point of view, $\mathbb{C}$ under addition and non zero elements of $\mathbb{C}$ under multiplication form groups. Along with the circle group $\mathbb{S}^1$, are some well-known examples of groups.

A subset H of a group G is a *subgroup* if it has the following properties:

- *Closure:* If a and b are in H, then ab is in H.
- 1 is in H.
- If a is in H, then a^{-1} is in H

Another important group which will come in our discourse is the notion of *quotient group.*

Given a group G and a normal subgroup N of G, the *quotient group* (or *factor group*) G/N, which is defined as the set of left cosets of N in G, that is,

$$G/N = \{\, gN \mid g \in G \,\}.$$

The group operation on G/N is given by $(gN)(hN) = (gh)N$ is well defined because N is normal in G.

Since *quotient topology* comes under our study, we will define it briefly for completion.

Definition 1.2 (quotient map)**.** *Let X and Y be topological spaces; let $p : X \to Y$ be surjective map. The map p is said to be* quotient map *provided a subset U of Y is open in Y iff $p^{-1}(U)$ is open in X*

The notion of quotient map is used to construct a topology on a set.

Definition 1.3. *If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology τ on A relative to which p is a quotient map; it is called the **quotient topology** induced by p*

2. Invariant Measure

We define measure on a space X (metric space, topological space, or could even be manifold) as follows:

Definition 2.1. *A measure μ on X is a set function on X taking values in $\mathbb{R}$ such that,*

i. $\mu \geq 0$, i.e., μ is positive definite for $A \subset X$ and $\mu(\phi) = 0$.

ii. μ is sub additive. That is,

$$\mu(\cup A) \leq \sum \mu(A), \qquad A \in X$$

and

$$\mu(\cup A) = \sum \mu(A),$$

if the union is pairwise disjoint.

A measurable space is a pair $(X, \mathcal{B})$, where $\mathcal{B}$ is a sigma field for the collection of subset of X satisfying the properties, $A' \in \mathcal{B}$ whenever $A \in \mathcal{B}$, $\cup A$ is in $\mathcal{B}$ for $A \in \mathcal{B}$, arbitrary union of members of $\mathcal{B}$ is closed and if A_1,$A_2 \in \mathcal{B}$ then $\cap_{i=1}^{n} A_i \in \mathcal{B}$ finite intersection closed on $(X, \mathcal{B})$. A measure μ makes X a measurable space and we write it as $(X, \mathcal{B}, \mu)$. Moreover, a probability measure means $\mu(X) = 1$.

To know what an invariant measure is, we introduce a transformation on X which is nothing but dynamical system associated with X. For example, consider $X = \mathbb{R}/\mathbb{Z}$, the circle group (or 1-dimensional torus) $\mathbb{T}(= \mathbb{S}^1)$. Its members are cosets of the form $r + \mathbb{Z}$, for $r \in \mathbb{R}$. Let us agree that a measure on X is to assign a number to every continuous function $f \in C(X)$, namely its integral $\int f(x)d\mu(x)$, or just $\int f d\mu$ with respect to μ.

Further, we want the following properties to hold, for $f \to \int f d\mu$ to be linear, and that $f \geq 0$ implies that $\int f d\mu \geq 0$. For the measure μ to be probability measure, we want that the integral of the constant function 1_X to be one. The Riemann integral $\int_0^1 f(r + \mathbb{Z})dr$ for $f \in C(\mathbb{R}/\mathbb{Z})$, in this regard, will be a probability measure. Also called the Lebesgue measure m_T [4].

To this end, suppose $T : X \to X$ be a continuous map. A probability measure μ on X is an invariant if,

$$\int f d\mu = \int f \circ T d\mu$$

Remark 2.2. *One should think of T as time evolution of the dynamical system, f to be the outcome of the physical experiment, and the integral as the expected value for the outcome of f. Then, the invariance of μ is simply the requirement that the experimental value of the outcome is same now and one unit time later.*

Remark 2.3. *The set $\mathcal{M}(T)$ of the invariant probability measure is critically dependent on the transformation T. For many maps T, this set is very large and it is impossible to give a reasonable description. However, sometimes we also have rigidity of invariant measures: the set of invariant measures shows a surprising amount of structure.*

3. Stability related ideas

In the study of dynamical systems, stability issues around fixed points (or invariant sets) arise quite often. For instance, the system of differential equations,

$$\frac{dz}{dt} = f(z) \tag{1}$$

for $z \in \mathbb{C}^n$ and $f : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic (analytical) with $f(0) = 0$ satisfies $z(t) = 0$ as a solution. This solution is said to be stable (past and future) if points near the origin remain near the origin under transformation. More aptly, we say that $z = 0$ is stable if for every neighborhood U of 0, there exists a neighborhood V with $0 \in V \subset U$ such that $z(0) \in V$ implies $z(t) \in U$ for all $t \in \mathbb{R}$.

A well known theorem due to Lyapunov [6] says that a necessary condition for the future stability of $z = 0$ is that the eigen values of Df_0 have non-positive real part, for the past stability they must have non-negative real part. Thus, for $z = 0$ to be stable the eigen values must be purely imaginary. Also see [1] for the stability of (1) for $z = 0$.

The following theorem clarifies the stability issues of (1) :

Theorem 3.1. *The solution $z = 0$ of (1) is stable if and only if there exists a holomorphic transformation $g : \mathbb{C}^n \to \mathbb{C}^n$, taking $z \mapsto \zeta$ with $g(0) = 0$ and $Dg_0 = I$ which brings (1) to the system if differential equations,*

$$\frac{d\zeta}{dt} = A\zeta \tag{2}$$

where A is the Jacobian of f at 0 and A is diagonalisable with purely imaginary eigenvalues.

Another interesting way of noticing (1) under the framework (we mean stability of solutions of the system of equations), by regarding the equation giving rise to integral curves of the holomorphic vector fields,

$$X = \sum f_k(z) \frac{\partial}{\partial z_k} \tag{3}$$

Let $\mathcal{M}$ be the set of holomorphic vector field which vanish at the origin, and let $\mathcal{G}$ be the group of holomorphic transformations which vanish at the origin, with $Dg_0 = I$. Then, there is a natural action $\mathcal{G} \times \mathcal{M} \to \mathcal{M}$ of $\mathcal{G}$ on $\mathcal{M}$ given by

$$(g, X) \mapsto gX$$
$$\text{where} \quad gX(z) = g_\star X(g^{-1}(z))$$

and $g_\star$ is the induced map on the tangent bundle.

In $\mathcal{M}$ two vector fields X, Y are said to be equivalent if,

$$X = gY \text{ for some } g \in \mathcal{G}$$

Now, given a vector field $X \in \mathcal{M}$, the linear part of X is given by

$$(X)(\text{tangential part}) = \sum_{j,k=1}^{n} a_{j,k} z_j \frac{\partial}{\partial z_k} \tag{4}$$

where

$$a_{j,k} = \frac{\partial f_k}{\partial z_j}(0)$$

Under this setting, the equation (1) can be expressed in the form (2), is same as that there exists a g such that

$$Ag(z) = A\zeta = \frac{d\zeta}{dt} = \frac{dg}{dt}(z) = g_\star \frac{dz}{dt} = g_\star f(z)$$

i.e., such that

$$A\zeta = g_\star f(g^{-1}(\zeta))$$

is condition that the corresponding vector field X whose linear part is (X).

In summary, the stability of equation (1) is closely connected to the theory of normal forms of holomorphic vector fields and in particular with the structure of the orbits of linear vector fields under the action of $\mathcal{G}$. It is very difficult to assert a necessary and sufficient condition which determines when a vector field is equivalent to its linear part. Indeed the conditions required are very delicate in the situation we are most interested, when its eigen values are purely imaginary.

4. On Badly Approximable Numbers

Motivation to study them comes from Diophantine approximations. It may be recalled that the theory of Diophantine equation is concerned with the solving equation of integer solutions as encountered by Diophantine of Alexandria around 3rd century CE. Unfortunately, this requirement for integer solutions falls short in certain equations of interest. In the attempt to overcome this situation and see that the requirement of integer solution exist is equivalent that rational solution exists. For instance, the polynomial function $f(x, y, z) = 0$ given by $x^2 + y^2 - z^2 = 0$ admits integer solution and this is an equivalent to

$$\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1 = 0$$

admitting rational solutions.

Such curves were seen as Fermat's curves (unit circle in $\mathbb{R}^2$ is Fermat's curve).

Bad news: If both fails, this leads to irrationals and their approximations (either for good or bad).

Definition 4.1. *A real number α is said to be badly approximable if there exists $\delta > 0$ such that,*

$$\left|\alpha - \frac{p}{q}\right| > \frac{\delta}{q^2} \tag{5}$$

for $p, q \in \mathbb{Z}, q > 0$.

We denote the set of badly approximable numbers by **B**.

Since every real number is a limiting sequence of rationals, such a **B** is reasonable to anticipate.

S. G. Dani, in his paper [2], refers to such numbers that interest some results on bounded geodesics, bounded orbits and similar topics, with the purpose of highlighting certain results explaining underlying ideas [9].

Their sets are of measure (Lebeague measure) zero, but do matter for their Hausdorff measure with respect to the usual metric is one, which is maximum possible for any subset of $\mathbb{R}$

This result due to V. Jarnik, [8], was strengthened by W. M. Schmidt who showed it to be a winning set for a certain game which implied in particular for an open in interval I (given) and a sequence of functions $f_1, f_2, \dots$ on I satisfying certain conditions and the set of α in I such that $f_i(\alpha) \in \mathbf{B}$.

5. Application of Measure Rigidity

Group actions on homogeneous spaces and classification of invariant measures have some interesting implications and applications [5]. Here, the group that acts on homogeneous spaces such as $SL_n(\mathbb{R})$ is a simple non-compact subgroup that is generated by unipotent one-parameter subgroups. Moreover, they are seen as the image of the homomorphism

$$\mathcal{U} : \mathbb{R} \to SL_n(\mathbb{R})$$

given by,

$$t \mapsto \mathcal{U}(t) = \exp(t_m) \qquad \text{for t } \in \mathbb{R}$$

for $m \in \mathrm{Mat}_n(\mathbb{R})$, m is nilpotent.

For instance,

$$\mathcal{U}(t) = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{\frac{-t}{2}} \end{pmatrix} \qquad \mathrm{diag}(e^{\frac{t}{2}}, e^{\frac{-t}{2}})$$

Hence,

$$X_n = SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})$$

turns out to be the subgroup, H of $SL_n(\mathbb{R})$ and then group action [3] on X_n by their right action.

To prove long standing Oppenheim conjecture [7] by Furstenberg and Margulis.

We also came to know the following conjectures are open:

Conjecture 1. *Let $n \geq 3$ and let*

$$A = \{diag(a_1, \dots, a_n) : a_1, \dots, a_n > 0, a_1 \cdots a_n = 1\}.$$

Then any $x \in X_n = SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})$ for which xA has compact closure in X_n must actually belong to a periodic (i.e. compact) orbit.

Conjecture 2. *Let $n \geq 3$ and A be as above. Then any A-invariant and ergodic probability measure on X_n is necessarily the normalized Haar measure on a finite volume orbit xH of an intermediate group $A \subseteq H \subseteq SL_n(\mathbb{R})$.*

Conjecture 3. *Let μ be an invariant and ergodic probability measure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ for the joint action of $x \mapsto 2x$ and $x \mapsto 3x$. Then either μ equals the Lebesgue measure or must have finite support (consisting of rational numbers).*

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